Lecture 2: Preliminaries

Real Numbers -

$\mathbb{R} = (-\infty, \infty)$ is a "Complete" space, which roughly means that if a sequence converges, its limit exists

e.g.) $\lim_{n \to \infty} \frac{1}{n} = 0$.

- This is stated via an axiom which says that any set bounded above has a least upper bound called a supremum.

  e.g.) ~~For $n \in \mathbb{N}$~~, $\{ \frac{1}{n} : n \geq 1 \}$ is bounded above by 1 since $\frac{1}{n} \leq 1$ for $n \geq 1$. Then, it has a supremum (namely 1)

  ~ this supremum is the smallest number greater than or equal to every number of the set. For example, the supremum of $(0,1)$ is 1, since for all $x \in (0,1)$ $x < 1$ implies $x \leq 1$ (so 1 is an upper bound) and this isn't true for any $y < 1$.

  ~ this is similar to the concept of a maximum. we also have an infimum or "greatest lower bound" similar to a minimum. The maximum of a set, however, must be included in the set. (the set $(0,1)$ has no maximum)

Domains in $\mathbb{R}^n$ -

$\mathbb{R}^n$ is the real vector space of dimension n. we denote length by the dot product

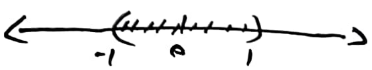
$$|x| = \sqrt{x \cdot x}$$

and denote $\lim_{k \to \infty} x_k = w$ if $\lim_{k \to \infty} |x_k - w| = 0$.

Since derivatives only care about local behavior, we focus on what "local" means

- a ball of radius $R > 0$ centered at $x_0 \in \mathbb{R}^n$ is the set $B(x_0, R) = \{ x \in \mathbb{R}^n ; |x - x_0| < R \}$

  ~> in $\mathbb{R}$, $B(0,1)$ looks like



  in $\mathbb{R}^2$, $B(0,1)$ is

- the boundary of a ball is the "edge"

$$\partial B(x_0, R) = \{ x \in \mathbb{R}^n ; |x - x_0| = R \}$$

- in general, we focus on a set A. This set is a neighborhood of x_0 if there is some r so

$$\mathcal{B}(x_0, r) \subset A$$

↳ A subset of, or contained in

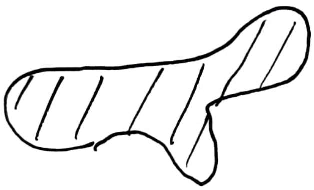
and the boundary ∂A is the set of points such that any ball around $x \in \partial A$ touches both A and something not in A. A set is closed if it contains its boundary.

① A is <u>open</u> if it is a neighborhood of any point in A.

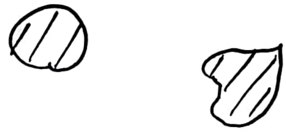
$(0,1)$ is open $[0,1]$ isn't

② $A \subset \mathbb{R}^n$ is <u>connected</u> if any two points of A may be joined by a line (continuous path) contained in the set

Connected



Not Connected



A domain U is an open, connected subset of $\mathbb{R}^n$.

<u>Differentiability</u>

- As above, we denote by $C^m(U)$ the m-times continuously differentiable functions on a domain U. Since U is open, ∂U doesn't overlap with U $(\partial U \cap U = \emptyset)$, and this is well-defined.

- Given a set V, we may define its <u>closure</u> $\bar{V} = V \cup \partial V$. (Show or explain why $\bar{V}$ is closed). We use this next.

- For a continuous function f, or $f \in C^0(U)$, we define its

  <u>Support</u> $\quad \text{supp}(f) = \overline{\{x \in U : f(x) \neq 0\}}$

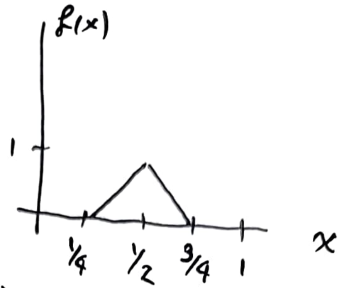
  this is the set "where f is interesting"

  –▷ We often focus on functions with compact support.
  A set is <u>compact</u> if it is closed and bounded
  (bounded means it may be put in some large ball around 0)
  We denote this by adding a C: $\quad C_c^m(U)$

- Lastly, a <u>smooth</u> function has derivatives of all orders. We denote the space of functions smooth on U to be $C^\infty(U)$.

e.g.) 1.) $\frac{1}{x} \in C^\infty((0,1))$ $\qquad$ $\frac{1}{x} \notin C^0([0,1])$

2.) Let $f(x)$ be



then $f(x) \in C^0((0,1))$. ~~Since~~
$\mathrm{Supp}(f) = [1/4, 3/4]$ so $f \in C_c^0((0,1))$.

3.) There exist smooth, compactly supported functions!

Consider

$$h(x) = \begin{cases} e^{\frac{-1}{1-x^2}} & |x|<1 \\ 0 & |x| \geq 1 \end{cases}$$

$$h'(x) = \begin{cases} \left(e^{\frac{-1}{1-x^2}}\right)\left(\frac{1}{(1-x^2)^2}\right)(-2x) & |x|<1 \\ 0 & |x| \geq 1 \end{cases}$$

use L'Hopital for endpoints

$\vdots$

$$h^m(x) = \begin{cases} \frac{q_m(x)}{(1-x^2)^m} e^{-\frac{1}{1-x^2}} & |x|<1 \\ 0 & |x| \geq 1 \end{cases}$$

- We can manipulate this to obtain step or bump functions $\chi_A(x) \in C_c^\infty(\mathbb{R}^n)$ which are 1 on any compact set $A \subseteq \mathbb{R}^n$ and 0 outside of a bounded open set containing A.

- Occasionally, we will need the boundary of our domain $\mathcal{U}$ to be nice - usually piecewise C^1 for integration-by-parts. What this means is that $\partial\mathcal{U}$ may be cut into pieces $A_1, \ldots A_n$ (a finite number) so that each A_i is the graph of a C^1 function. We will usually assume this without a need for deeper understanding.